

AN EXPLICIT STRUCTURE OF THE GRADED RING OF MODULAR FORMS OF SMALL LEVEL

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1. INTRODUCTION

For each integers $k \geq 0$ and $N \geq 1$, let $\mathcal{M}_k(N)$ be the space of all modular forms of weight k with respect to the congruence subgroup

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \in N\mathbb{Z} \right\}$$

of level N (cf [2, Definition 1.2.3]), and

$$\mathcal{M}(N) = \bigoplus_{k=0}^{\infty} \mathcal{M}_k(N)$$

be the graded ring of modular forms for $\Gamma_0(N)$. When $N = 1$, it is well-known that, as a \mathbb{C} -algebra, $\mathcal{M}(1)$ is generated by Eisenstein series E_4 and E_6 of weight 4 and 6, and these two forms are algebraically independent:

$$\mathcal{M}(1) \simeq \mathbb{C}[E_4, E_6].$$

For each N , we note that, $\mathcal{M}(N)$ is generated by finitely many modular forms (cf [1]), however, it is not necessarily isomorphic to the polynomial ring. For instance, when $N = 3$, we prove

$$\mathcal{M}(3) \simeq \mathbb{C}[C_3, \alpha_3, \beta_3] / (\alpha_3^2 - C_3\beta_3)$$

for some $C_3 \in \mathcal{M}_2(3)$, $\alpha_3 \in \mathcal{M}_4(3)$ and $\beta_3 \in \mathcal{M}_6(3)$ (cf. Theorem 2). The aim of this paper is to give the ring structure of $\mathcal{M}(N)$ explicitly for

$$N = 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 16, 18, 25.$$

The method we use is summarized as follows: First, for each N , we take some suitable forms f_1, \dots, f_h from $\mathcal{M}(N)$. Then, for each k , we see that a basis $\{b_1, \dots, b_d\}$ ($d = \dim \mathcal{M}_k(N)$) of $\mathcal{M}_k(N)$ is obtained by f_i 's (cf. §3), thus, the natural homomorphism from the polynomial ring $\mathbb{C}[f_1, \dots, f_h]$ to $\mathcal{M}(N)$ is surjective. Second, in §4 and 5, we show some relations between f_i 's, i.e., give some elements of the kernel of the above-mentioned homomorphism. Third, using the result in §6, we calculate the Hilbert functions, that is, generating series of the dimensions, and we obtain the isomorphism in §7. In this context, we may regard $\mathcal{M}(N)$ as a subring of $\mathbb{C}[[q]]$, where $q = e^{2\pi iz}$ ($z \in \mathbb{C}$, $\mathrm{Im} z > 0$), via the Fourier expansion. Our basis $\{b_1, \dots, b_d\}$ satisfy that $b_j \in q^{j-1} + \mathbb{C}[[q]]q^j$ for each j (cf. §3), hence we see

$$\mathcal{M}_k(N) \cap \mathbb{C}[[q]]q^d = \{0\},$$

which is similar to a result of Sturm [4]. This property will be used for the proof of relations between modular forms.

It should be emphasized that, when $N \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 16, 18, 25\}$, $\mathcal{M}(N)$ is generated by some Eisenstein series of weight 2 or 4 or 6, and f_i 's are

obtained by such forms. We shall review the Eisenstein series and its Fourier expansion in §2. Moreover, we show $f_i \in \mathbb{Z}[[q]]$ for each i , and

$$b_j \in q^{j-1} + \mathbb{Z}[[q]]q^j$$

for each j (cf. §8). When $N = 1$, such an integral basis was also taken by Lang [3, Ch.X, Theorem 4.4].

2. EISENSTEIN SERIES

For each even integer $k > 0$, let B_k be the k -th Bernoulli number, $\sigma_k(n) = \sum_{d|n} d^k$ and

$$E_k = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n$$

the Eisenstein series of weight k . It is well-known that if $k \geq 4$, $E_k \in \mathcal{M}_k(1)$. In particular, the following three forms will play important roles in the sequel:

$$E_2 = 1 - 24 \sum_n \sigma_1(n)q^n,$$

$$E_4 = 1 + 240 \sum_n \sigma_3(n)q^n,$$

$$E_6 = 1 - 504 \sum_n \sigma_5(n)q^n.$$

For each $h \in \mathbb{N}$, we note that $\mathcal{M}(N) \subset \mathcal{M}(Nh)$. For each $f \in \mathcal{M}_k(N)$, we define $f^{(h)} \in \mathcal{M}_k(Nh)$ to be

$$f^{(h)}(q) = f(q^h).$$

For each $N > 1$, we put

$$C_N = \frac{1}{(N-1,24)}(NE_2^{(N)} - E_2),$$

then we see $C_N \in \mathcal{M}_2(N)$ (cf. [2, Exercises 1.2.8]), and

$$C_N = \frac{N-1}{(N-1,24)} + \frac{24}{(N-1,24)} \sum_n \tau_N(n)q^n$$

where $\tau_N(n) = \sum_{d|n, N \nmid d} d$. In addition, we see

$$\begin{aligned} C_N^{(h)} &= \frac{1}{(N-1,24)}(NE_2^{(Nh)} - E_2^{(h)}) \\ &= \frac{1}{(N-1,24)h}(- (h-1, 24)C_h + (Nh-1, 24)C_{Nh}). \end{aligned}$$

For each prime number p , we put

$$\alpha_p = \frac{1}{240}(E_4 - E_4^{(p)}),$$

then we see $\alpha_p \in \mathcal{M}_4(p) \cap (1 + \mathbb{Z}[[q]]q)$.

For a primitive Dirichlet character $\chi \bmod N$, put $\sigma_\chi(n) = \sum_{d|n} \chi(d)d$ and

$$E_\chi = \sum_n \sigma_{\chi^2}(n)\overline{\chi}(n)q^n,$$

then we have $E_\chi \in \mathcal{M}_2(N^2)$ (cf. [2, §4.5 and 4.6]). For $N = 3, 4, 5$ we denote by ρ_N the non-trivial real character mod N . Moreover let χ_5 be the Dirichlet character

mod 5 such that $\chi_5(2) = \sqrt{-1}$, and

$$\begin{aligned} E_{r_5} &= \frac{1}{2}(E_{\chi_5} + E_{\overline{\chi_5}}), \\ E_{i_5} &= \frac{1}{2i}(E_{\chi_5} - E_{\overline{\chi_5}}). \end{aligned}$$

Then, since $(\chi_5)^2 = \rho_5$ and $(\chi_5)^3 = \overline{\chi_5}$, we see

$$\begin{aligned} E_{r_5} &= \sum_{n \equiv 1 \pmod{5}} \sigma_{\rho_5}(n)q^n - \sum_{n \equiv 4 \pmod{5}} \sigma_{\rho_5}(n)q^n, \\ E_{i_5} &= - \sum_{n \equiv 2 \pmod{5}} \sigma_{\rho_5}(n)q^n + \sum_{n \equiv 3 \pmod{5}} \sigma_{\rho_5}(n)q^n. \end{aligned}$$

3. BASIS OF $\mathcal{M}_k(N)$

For each k and N , we would take a basis $\{b_1, \dots, b_d\}$ ($d = \dim \mathcal{M}_k(N)$) of $\mathcal{M}_k(N)$, such that

$$b_i \in q^{i-1} + \mathbb{C}[[q]]q^i \quad (1 \leq i \leq d).$$

First, we write down the following dimension formulas for even $k \geq 0$:

$$\begin{aligned} \dim \mathcal{M}_k(1) &= \left[\frac{k}{12} \right] + 1 - \delta_{12\mathbb{Z}+2}(k), \\ \dim \mathcal{M}_k(2) &= \left[\frac{k}{4} \right] + 1, \\ \dim \mathcal{M}_k(3) &= \left[\frac{k}{3} \right] + 1, \\ \dim \mathcal{M}_k(4) &= \frac{k}{2} + 1, \\ \dim \mathcal{M}_k(5) &= 2 \left[\frac{k}{4} \right] + 1, \\ \dim \mathcal{M}_k(6) &= k + 1, \\ \dim \mathcal{M}_k(7) &= 2 \left[\frac{k}{3} \right] + 1, \\ \dim \mathcal{M}_k(8) &= k + 1, \\ \dim \mathcal{M}_k(9) &= k + 1, \\ \dim \mathcal{M}_k(10) &= k + 2 \left[\frac{k}{4} \right] + 1, \\ \dim \mathcal{M}_k(12) &= 2k + 1, \\ \dim \mathcal{M}_k(16) &= 2k + 1, \\ \dim \mathcal{M}_k(18) &= 3k + 1, \\ \dim \mathcal{M}_k(25) &= 2k + 2 \left[\frac{k}{4} \right] + 1, \end{aligned}$$

where $[]$ denotes the Gauss symbol and

$$\delta_X(k) = \begin{cases} 1 & \text{if } k \in X, \\ 0 & \text{if } k \notin X \end{cases}$$

(cf [2, Theorem 3.5.1]).

When $N = 1$, let

$$\Delta = \frac{1}{12^3}(E_4^3 - E_6^2) \in \mathcal{M}_{12}(1)$$

be the Ramanujan Δ -function, then for each $l > 0$ we see $\mathcal{M}_{12l}(1) \supset \sum_{i=0}^l \mathbb{C}E_4^{3(l-i)}\Delta^i = \bigoplus_{i=0}^l \mathbb{C}E_4^{3(l-i)}\Delta^i$, and therefore, comparing the dimensions on both sides induces

$$\mathcal{M}_{12l}(1) = \bigoplus_{i=0}^l \mathbb{C}E_4^{3(l-i)}\Delta^i.$$

Similarly, we have

$$\begin{aligned}\mathcal{M}_{12l+4}(1) &= \mathcal{M}_{12l}(1)E_4, \\ \mathcal{M}_{12l+6}(1) &= \mathcal{M}_{12l}(1)E_6, \\ \mathcal{M}_{12l+8}(1) &= \mathcal{M}_{12l}(1)E_4^2, \\ \mathcal{M}_{12l+10}(1) &= \mathcal{M}_{12l}(1)E_4E_6, \\ \mathcal{M}_{12l+14}(1) &= \mathcal{M}_{12l}(1)E_4^2E_6.\end{aligned}$$

For each $N > 1$ we shall take, in the rest of this section, some modular forms and represent the basis $\{b_j\}$ by such forms.

3.1. The case $N=4,6,8,9,12,16,18$. In these cases, we have seen that

$$\dim \mathcal{M}_k(N) = s\frac{k}{2} + 1$$

for even $k \geq 0$, where $s = \dim \mathcal{M}_2(N) - 1$. We take a $(s+1)$ -tuple (f_0, f_1, \dots, f_s) satisfying

$$f_i \in \mathcal{M}_2(N) \cap (q^i + \mathbb{C}[[q]]q^{i+1}) \quad (i = 0, 1, \dots, s).$$

Then we have

$$\mathcal{M}_{2l}(N) = \bigoplus_{i=0}^l \mathbb{C}f_0^{l-i}f_1^i \oplus \bigoplus_{i=1}^l \mathbb{C}f_1^{l-i}f_2^i \oplus \dots \oplus \bigoplus_{i=1}^l \mathbb{C}f_{s-1}^{l-i}f_s^i.$$

Indeed, for a given N , such tuple can be taken as follows:

When $N = 4$, $(f_0, f_1) = (C_4, \alpha_4)$, where

$$\alpha_4 = \frac{1}{16}(C_2 - C_4).$$

When $N = 6$, $(f_0, f_1, f_2) = (C_3^{(2)}, \alpha_6, \beta_6)$, where

$$\begin{aligned}\alpha_6 &= \frac{1}{12}(C_2 - C_3), \\ \beta_6 &= \frac{1}{12}(C_3^{(2)} - C_2^{(3)}).\end{aligned}$$

When $N = 8$, $(f_0, f_1, f_2) = (C_4^{(2)}, \alpha_4, \alpha_4^{(2)})$.

When $N = 9$, $(f_0, f_1, f_2) = (C_3, E_{\rho_3}, \beta_9)$, where

$$\beta_9 = \frac{1}{6}(\frac{1}{9}(C_3 - C_9) - E_{\rho_3}).$$

When $N = 12$, $(f_0, f_1, f_2, f_3, f_4) = (C_3^{(2)}, \alpha_6, \beta_6, \alpha_4^{(3)}, \beta_6^{(2)})$.

When $N = 16$, $(f_0, f_1, f_2, f_3, f_4) = (C_4^{(2)}, \alpha_4, \alpha_4^{(2)}, \gamma_{16}, \alpha_4^{(4)})$, where

$$\gamma_{16} = \frac{1}{8}(\alpha_4 - E_{\rho_4}).$$

When $N = 18$, $(f_0, f_1, f_2, f_3, f_4, f_5, f_6) = (C_9^{(2)}, \alpha_6, \beta_6, \alpha_6^{(3)}, \beta_9^{(2)}, \epsilon_{18}, \beta_6^{(3)})$, where

$$\epsilon_{18} = \frac{1}{2}(\beta_9 - E_{\rho_3}^{(2)} - 3\beta_9^{(2)}) + \beta_6^{(3)}.$$

3.2. The case $N=2,5,10,25$. In these cases, we have seen that

$$\dim \mathcal{M}_k(N) = s \frac{k}{2} + t \left\lceil \frac{k}{4} \right\rceil + 1,$$

where $s = \dim \mathcal{M}_2(N) - 1$ and $t = \dim \mathcal{M}_4(N) - 2s - 1$. We take a $(s+t+1)$ -tuple $(f_0, f_1, \dots, f_s; g_1, g_2, \dots, g_t)$ satisfying

$$\begin{aligned} f_i &\in \mathcal{M}_2(N) \cap (q^i + \mathbb{C}[[q]]q^{i+1}) \quad (i = 0, 1, \dots, s), \\ g_i &\in \mathcal{M}_4(N) \cap (q^{2s+i} + \mathbb{C}[[q]]q^{2s+i+1}) \quad (i = 1, \dots, t). \end{aligned}$$

Then we have

$$\begin{aligned} \mathcal{M}_{4l}(N) &= \bigoplus_{i=0}^{2l} \mathbb{C} f_0^{2l-i} f_1^i \oplus \bigoplus_{i=1}^{2l} \mathbb{C} f_1^{2l-i} f_2^i \oplus \dots \oplus \bigoplus_{i=1}^{2l} \mathbb{C} f_{s-1}^{2l-i} f_s^i \\ &\quad \oplus \bigoplus_{i=0}^l \mathbb{C} f_s^{2(l-i)} g_1^i \oplus \bigoplus_{i=1}^l \mathbb{C} g_1^{l-i} g_2^i \oplus \dots \oplus \bigoplus_{i=1}^l \mathbb{C} g_{t-1}^{l-i} g_t^i, \\ \mathcal{M}_{4l+2}(N) &= \mathcal{M}_{4l}(N) f_0 \oplus \mathbb{C} g_t^l f_1 \oplus \dots \oplus \mathbb{C} g_t^l f_s. \end{aligned}$$

Indeed, such tuple can be taken as follows:

When $N = 2$, $(f_0; g_1) = (C_2; \alpha_2)$.

When $N = 5$, $(f_0; g_1, g_2) = (C_5; \alpha_5, \beta_5)$, where

$$\beta_5 = \frac{1}{36}(-C_5^2 + 12\alpha_5 + E_4^{(5)}).$$

When $N = 10$, $(f_0, f_1, f_2; g_1, g_2) = (C_2, \alpha_{10}, \beta_{10}; \alpha_2^{(5)}, \zeta_{10})$, where

$$\begin{aligned} \alpha_{10} &= \frac{1}{8}(C_2 - 4C_5 + C_{10}), \\ \beta_{10} &= \frac{1}{6}(C_5^{(2)} - C_2^{(5)}), \\ \zeta_{10} &= \frac{1}{4}(\beta_{10}^2 - \beta_5^{(2)}). \end{aligned}$$

When $N = 25$, $(f_0, f_1, f_2, f_3, f_4; g_1, g_2) = (C_5, E_{\rho_5}, E_{i_5}, \gamma_{25}, \delta_{25}; \iota_{25}, \beta_5^{(5)})$, where

$$\begin{aligned} \gamma_{25} &= \frac{1}{10}(-E_{\rho_5} + E_{r_5} - 3E_{i_5}), \\ \delta_{25} &= \frac{1}{100}(C_5 - C_{25} + 5E_{\rho_5} - 10E_{r_5}), \\ \iota_{25} &= \alpha_5^{(5)} + (-E_{r_5} + \gamma_{25} - 2\delta_{25})\delta_{25} - \beta_5^{(5)}. \end{aligned}$$

3.3. The case $N=3,7$. In these cases, we have seen that

$$\dim \mathcal{M}_k(N) = s \left\lceil \frac{k}{3} \right\rceil + 1,$$

where $s = \dim \mathcal{M}_4(N) - 1$. We take a $(2s+1)$ -tuple $(f_0; g_1, \dots, g_s; h_1, \dots, h_s)$ satisfying

$$\begin{aligned} f_0 &\in \mathcal{M}_2(N) \cap (1 + \mathbb{C}[[q]]q), \\ g_i &\in \mathcal{M}_4(N) \cap (q^i + \mathbb{C}[[q]]q^{i+1}) \quad (i = 1, \dots, s), \\ h_i &\in \mathcal{M}_6(N) \cap (q^{s+i} + \mathbb{C}[[q]]q^{s+i+1}) \quad (i = 1, \dots, s). \end{aligned}$$

Then we have

$$\begin{aligned}\mathcal{M}_{6l}(N) &= \bigoplus_{i=0}^l \mathbb{C} f_0^{3l-2i} g_1^i \oplus \bigoplus_{i=1}^l \mathbb{C} f_0^l g_1^{l-i} g_2^i \oplus \cdots \oplus \bigoplus_{i=1}^l \mathbb{C} f_0^l g_{s-1}^{l-i} g_s^i \\ &\quad \oplus \bigoplus_{i=1}^l \mathbb{C} (f_0 g_s)^{l-i} h_1^i \oplus \bigoplus_{i=1}^l \mathbb{C} h_1^{l-i} h_2^i \oplus \cdots \oplus \bigoplus_{i=1}^l \mathbb{C} h_{s-1}^{l-i} h_s^i, \\ \mathcal{M}_{6l+2}(N) &= \mathcal{M}_{6l}(N) f_0, \\ \mathcal{M}_{6l+4}(N) &= \mathcal{M}_{6l}(N) f_0^2 \oplus \mathbb{C} h_s^l g_1 \oplus \cdots \oplus \mathbb{C} h_s^l g_s.\end{aligned}$$

Indeed, such tuple can be taken as follows:

When $N = 3$, $(f_0; g_1; h_1) = (C_3; \alpha_3; \beta_3)$, where

$$\beta_3 = \frac{1}{12} \left(\frac{1}{504} (E_6^{(3)} - E_6) - C_3 \alpha_3 \right).$$

When $N = 7$, $(f_0; g_1, g_2; h_1, h_2) = (C_7; \alpha_7, \beta_7; \gamma_7, \delta_7)$, where

$$\begin{aligned}\beta_7 &= \frac{1}{32} (-C_7^2 + 8\alpha_7 + E_4^{(7)}), \\ \gamma_7 &= \frac{1}{360} \left(\frac{29}{2} (C_7 E_4^{(7)} - E_6^{(7)}) + \frac{17}{504} (E_6^{(7)} - E_6) - 75C_7 \alpha_7 + 240C_7 \beta_7 \right), \\ \delta_7 &= \frac{1}{360} \left(\frac{7}{2} (C_7 E_4^{(7)} - E_6^{(7)}) + \frac{1}{504} (E_6^{(7)} - E_6) - 15C_7 \alpha_7 + 120C_7 \beta_7 \right).\end{aligned}$$

4. RELATIONS BETWEEN MODULAR FORMS FOR N DIVIDING 12

We define

$$O_3 = \alpha_3^2 - C_3 \beta_3,$$

$$O_6 = \alpha_6^2 - C_3^{(2)} \beta_6,$$

$$\begin{aligned}O_{12b} &= \gamma_{12}^2 - \beta_6 \beta_6^{(2)}, \\ O_{12c} &= C_3^{(2)} \gamma_{12} - (\beta_6 + 2\gamma_{12} + 4\beta_6^{(2)}) \alpha_6, \\ O_{12d} &= \alpha_6 \gamma_{12} - (\beta_6 + 2\gamma_{12} + 4\beta_6^{(2)}) \beta_6, \\ O_{12e} &= \alpha_6 \beta_6^{(2)} - (\beta_6 + 2\gamma_{12} + 4\beta_6^{(2)}) \gamma_{12}, \\ O_{12f} &= C_3^{(2)} \beta_6^{(2)} - \beta_6^2 - 4\alpha_6 \beta_6^{(2)} - 4\gamma_{12}^2 - 16\beta_6^{(2)2},\end{aligned}$$

where

$$\gamma_{12} = \alpha_4^{(3)} - \beta_6^{(2)}.$$

In this section, we give some relations between modular forms, in particular, show that all forms defined above are 0. For $N = 1$, we have seen that for each $k \geq 8$, E_k can be represented by E_4 and E_6 . For example, we get $E_8, E_4^2 \in \mathcal{M}_8(1) \cap (1 + \mathbb{C}[[q]]q)$, and thus $E_8 - E_4^2 \in \mathcal{M}_8(1) \cap \mathbb{C}[[q]]q = \{0\}$, that is

$$E_8 = E_4^2.$$

4.1. **The cases $\mathbf{N=4,6,12}$.** First, we have $O_6 \in \mathcal{M}_4(6) \cap \mathbb{C}[[q]]q^5 = \{0\}$ and $O_{12b} \in \mathcal{M}_4(12) \cap \mathbb{C}[[q]]q^9 = \{0\}$. We can show $O_{12c} = O_{12d} = O_{12e} = O_{12f} = 0$ in a similar fashion, we also give algebraic proofs of them. We see

$$\begin{aligned} C_2 &= C_3^{(2)} + 24\alpha_6 + 36\beta_6, \\ C_3 &= C_3^{(2)} + 12\alpha_6 + 36\beta_6, \\ C_4 &= C_3^{(2)} + 8\alpha_6 + 20\beta_6 + 16\gamma_{12} - 16\beta_6^{(2)}, \\ C_6 &= 5C_3^{(2)} + 24\alpha_6 + 36\beta_6, \\ C_{12} &= 11C_3^{(2)} + 24\alpha_6 - 36\beta_6 - 144\gamma_{12} - 144\beta_6^{(2)}, \end{aligned}$$

and

$$\begin{aligned} 0 &= -O_6^{(2)} \\ &= -\alpha_6^{(2)2} + C_3^{(4)}\beta_6^{(2)} \\ &= -\left(\frac{1}{12}\left(\frac{1}{2}(3C_4 - C_2) - C_3^{(2)}\right)\right)^2 + \frac{1}{8}(-3C_4 + C_{12})\beta_6^{(2)} \\ &= C_3^{(2)}\beta_6^{(2)} - \beta_6^2 - 4\beta_6\gamma_{12} - 8\beta_6\beta_6^{(2)} - 4\gamma_{12}^2 - 16\gamma_{12}\beta_6^{(2)} - 16\beta_6^{(2)2} \\ &= 8O_{12b} + 4O_{12e} + O_{12f}. \end{aligned}$$

We also see

$$\begin{aligned} \alpha_6\gamma_{12}O_{12f} &= (C_3^{(2)}\gamma_{12} + \alpha_6(\beta_6 + 4\beta_6^{(2)}) - 2\alpha_6\gamma_{12})O_{12e} - (\beta_6 + 2\gamma_{12} + 4\beta_6^{(2)})\beta_6^{(2)}O_6 \\ &\quad + (C_3^{(2)}\beta_6 + 2C_3^{(2)}\gamma_{12} + 4C_3^{(2)}\beta_6^{(2)} - 8\alpha_6\gamma_{12})O_{12b}, \end{aligned}$$

and thus

$$0 = (C_3^{(2)}\gamma_{12} + \alpha_6(\beta_6 + 4\beta_6^{(2)}) + 2\alpha_6\gamma_{12})O_{12e}.$$

We note that $\mathbb{C}[[q]]$ is a integral domain, and hence $0 = O_{12e} = O_{12f}$. Moreover we have $O_{12d} = \frac{\beta_6}{\gamma_{12}}O_{12e} = 0$ and $O_{12c} = \frac{C_3^{(2)}}{\alpha_6}O_{12d} = 0$.

In $\mathcal{M}(4)$ we get

$$E_4 = C_2^2 + 192C_4\alpha_4$$

since $E_4 - (C_2^2 + 192C_4\alpha_4) \in \mathcal{M}_4(4) \cap \mathbb{C}[[q]]q^3 = \{0\}$. In addition, considering in $\mathcal{M}(12)$, we get a relation in $\mathcal{M}(6)$:

$$\begin{aligned} E_4 &= C_2^2 + 12C_2C_4 - 12C_4^2 \\ &= C_3^{(2)2} + 240C_3^{(2)}\alpha_6 + 264C_3^{(2)}\beta_6 - 192C_3^{(2)}\gamma_{12} + 192C_3^{(2)}\beta_6^{(2)} \\ &\quad + 2112\alpha_6^2 + 7104\alpha_6\beta_6 + 1536\alpha_6\gamma_{12} - 1536\alpha_6\beta_6^{(2)} + 5136\beta_6^2 \\ &\quad - 768\beta_6\gamma_{12} + 768\beta_6\beta_6^{(2)} - 3072\gamma_{12}^2 + 6144\gamma_{12}\beta_6^{(2)} - 3072\beta_6^{(2)2} \\ &= C_3^{(2)2} + 240C_3^{(2)}\alpha_6 + 792C_3^{(2)}\beta_6 + 1584\alpha_6^2 + 6912\alpha_6\beta_6 + 6480\beta_6^2 \\ &\quad + 48(11O_6 - 112O_{12b} - 4O_{12c} + 24O_{12d} - 32O_{12e} + 4O_{12f}) \\ &= C_2^2 + C_2C_3 - C_3^2 + 5C_3C_6 - C_6^2. \end{aligned}$$

4.2. **The case N=2.** We see

$$\begin{aligned}
E_4^{(2)} &= (C_2^2 + C_2C_3 - C_3^2 + 5C_3C_6 - C_6^2)^{(2)} \\
&= C_3^{(2)2} + 216C_3^{(2)}\beta_6 + 432C_3^{(2)}\gamma_{12} + 288C_3^{(2)}\beta_6^{(2)} - 1152\beta_6^2 - 4608\beta_6\gamma_{12} \\
&\quad - 5760\beta_6\beta_6^{(2)} - 4608\gamma_{12}^2 - 11520\gamma_{12}\beta_6^{(2)} - 4608\beta_6^{(2)2} \\
&= C_3^{(2)2} - 108C_3^{(2)}\beta_6 + 324\alpha_6^2 + 432\alpha_6\beta_6 \\
&\quad + 36(-9O_6 + 64O_{12b} + 12O_{12c} + 24O_{12d} + 80O_{12e} + 8O_{12f}) \\
&= \frac{1}{4}(5C_2^2 - (C_2^2 + C_2C_3 - C_3^2 + 5C_3C_6 - C_6^2)) \\
&= \frac{1}{4}(5C_2^2 - E_4).
\end{aligned}$$

We note $E_4^{(2)} = C_2^2 - 48C_4\alpha_4$ and $\alpha_2 = \frac{1}{240}(E_4 - E_4^{(2)}) = C_4\alpha_4$.

We get

$$E_6 = C_2(4C_2^2 - 3E_4)$$

since $E_6 - C_2(4C_2^2 - 3E_4) \in \mathcal{M}_6(2) \cap \mathbb{C}[[q]]q^2 = \{0\}$. We also see

$$\begin{aligned}
E_6^{(2)} &= C_2^{(2)}(4C_2^{(2)2} - 3(C_2^2 - 48C_4\alpha_4)) \\
&= (C_4 - 8\alpha_4)(C_4 + 16\alpha_4)(C_4 - 32\alpha_4) \\
&= (C_4 + 16\alpha_4)(C_2^2 - 72\alpha_2) \\
&= \frac{1}{8}C_2(11C_2^2 - 3E_4),
\end{aligned}$$

since $C_2 = C_4 + 16\alpha_4$ and $C_2^{(2)} = C_4 - 8\alpha_4$.

4.3. **The case N=3.** We see

$$\begin{aligned}
E_4^{(3)} &= (C_2^2 + 12C_2C_4 - 12C_4^2)^{(3)} \\
&= C_3^{(2)2} - 24C_3^{(2)}\beta_6 + 192C_3^{(2)}\gamma_{12} + 192C_3^{(2)}\beta_6^{(2)} + 144\beta_6^2 - 2304\beta_6\gamma_{12} \\
&\quad - 2304\beta_6\beta_6^{(2)} - 3072\gamma_{12}^2 - 6144\gamma_{12}\beta_6^{(2)} - 3072\beta_6^{(2)2} \\
&= C_3^{(2)2} - 8C_3^{(2)}\beta_6 - 16\alpha_6^2 + 192\alpha_6\beta_6 + 720\beta_6^2 \\
&\quad + 16(O_6 + 48O_{12b} + 12O_{12c} + 24O_{12d} + 96O_{12e} + 12O_{12f}), \\
&= \frac{1}{9}(10C_3^2 - (C_2^2 + C_2C_3 - C_3^2 + 5C_3C_6 - C_6^2)) \\
&= \frac{1}{9}(10C_3^2 - E_4),
\end{aligned}$$

and

$$\begin{aligned}
E_6^{(3)} &= (C_2(4C_2^2 - 3E_4))^{(3)} \\
&= 4C_2^{(3)3} - 3C_2^{(3)}(C_3^{(2)2} - 8C_2^{(3)}\beta_6 - 16\alpha_6^2 + 192\alpha_6\beta_6 + 720\beta_6^2) \\
&= C_3^{(2)3} - 84C_3^{(2)2}\beta_6 + 48C_3^{(2)}\alpha_6^2 - 576C_3^{(2)}\alpha_6\beta_6 - 720C_3^{(2)}\beta_6^2 - 576\alpha_6^2\beta_6 \\
&\quad + 6912\alpha_6\beta_6^2 + 19008\beta_6^3 \\
&= C_3^{(2)3} + \frac{4}{3}C_3^{(2)2}\beta_6 - \frac{112}{3}C_3^{(2)}\alpha_6^2 - 64C_3^{(2)}\alpha_6\beta_6 - 720C_3^{(2)}\beta_6^2 - 512\alpha_6^3 - 576\alpha_6^2\beta_6 \\
&\quad + 6912\alpha_6\beta_6^2 + 19008\beta_6^3 + \frac{256}{3}(C_3^{(2)} + 6\alpha_6)O_6 \\
&= \frac{1}{27}(35C_3^3 - 7C_3E_4 - C_2(4C_2^2 - 3E_4)) \\
&= \frac{1}{27}(35C_3^3 - 7C_3E_4 - E_6).
\end{aligned}$$

Finally, since $\beta_6 = \alpha_6^2/C_3^{(2)}$, we have

$$C_3 = (C_3^{(2)} + 6\alpha_6)^2 / C_3^{(2)},$$

$$\begin{aligned}\alpha_3 &= \frac{1}{240}(E_4 - \frac{1}{9}(10C_3^2 - E_4)) \\ &= \frac{1}{6^3}(E_4 - C_3^2) \\ &= C_3^{(2)}\alpha_6 + 10\alpha_6^2 + 28\alpha_6\beta_6 + 24\beta_6^2 \\ &= (C_3^{(2)} + 2\alpha_6)^2(C_3^{(2)} + 6\alpha_6)\alpha_6 / C_3^{(2)2},\end{aligned}$$

$$\begin{aligned}\beta_3 &= \frac{1}{12}\left(\frac{1}{504}\left(\frac{1}{27}(35C_3^3 - 7C_3E_4 - E_6) - E_6\right) - C_3\frac{1}{6^3}(E_4 - C_3^2)\right) \\ &= \frac{1}{108^2}(7C_3^3 - 5C_3E_4 - 2E_6) \\ &= C_3^{(2)}\alpha_6^2 + 8\alpha_6^3 + 24\alpha_6^2\beta_6 + 32\alpha_6\beta_6^2 + 16\beta_6^3 \\ &= (C_3^{(2)} + 2\alpha_6)^4\alpha_6^2 / C_3^{(2)3},\end{aligned}$$

thus $\alpha_3^2 = C_3\beta_3$ i.e. $O_3 = 0$.

5. RELATIONS BETWEEN MODULAR FORMS FOR N NOT-DIVIDING 12

Put

$$\begin{aligned}\alpha_9 &= E_{\rho_3} + 9\beta_9, \\ u_{10} &= \frac{1}{3}(-2C_2 + 5C_5), \\ \epsilon_{10} &= \alpha_2^{(5)} - 5\zeta_{10}, \\ u_{18} &= C_9^{(2)} - 3\beta_6, \\ \alpha_{18} &= \alpha_6 + 3\beta_6, \\ \gamma_{18} &= \alpha_{18}^{(3)}, \\ \delta_{18} &= \beta_9^{(2)} - \epsilon_{18} + 2\beta_6^{(3)}, \\ u_{25} &= C_{25} - 5E_{r_5} - 25\delta_{25}, \\ \alpha_{25} &= E_{\rho_5} + 5\gamma_{25}.\end{aligned}$$

We define

$$\begin{aligned}O_5 &= \alpha_5^2 - (C_5^2 + 4\alpha_5 - 8\beta_5)\beta_5, \\ O_{7a} &= \beta_7^2 - C_7\delta_7, \\ O_{7b} &= C_7\gamma_7 - \alpha_7\beta_7, \\ O_{7c} &= \beta_7\gamma_7 - \alpha_7\delta_7, \\ O_{7d} &= \alpha_7^2 - (C_7^2 + 7\alpha_7 - 19\beta_7)\beta_7, \\ O_{7e} &= \alpha_7\gamma_7 - (C_7\beta_7 + 7\gamma_7 - 19\delta_7)\beta_7, \\ O_{7f} &= \gamma_7^2 - (C_7\beta_7 + 7\gamma_7 - 19\delta_7)\delta_7, \\ O_8 &= \alpha_4^2 - C_4^{(2)}\alpha_4^{(2)}, \\ O_9 &= \alpha_9^2 - C_3\beta_9,\end{aligned}$$

$$\begin{aligned}
O_{10a} &= \alpha_{10}^2 - u_{10}\beta_{10}, \\
O_{10b} &= \alpha_{10}\epsilon_{10} - u_{10}\zeta_{10}, \\
O_{10c} &= \beta_{10}\epsilon_{10} - \alpha_{10}\zeta_{10}, \\
O_{10d} &= \alpha_{10}\beta_{10}^2 - (u_{10} + 8\alpha_{10} + 20\beta_{10})\epsilon_{10}, \\
O_{10e} &= \beta_{10}^3 - (\alpha_{10}\epsilon_{10} + 8\beta_{10}\epsilon_{10} + 20\beta_{10}\zeta_{10}), \\
O_{10f} &= \beta_{10}^2\zeta_{10} - (\epsilon_{10}^2 + 8\epsilon_{10}\zeta_{10} + 20\zeta_{10}^2),
\end{aligned}$$

$$\begin{aligned}
O_{16b} &= \gamma_{16}^2 - \alpha_4^{(2)}\alpha_4^{(4)}, \\
O_{16c} &= C_4^{(2)}\gamma_{16} - (\alpha_4^{(2)} + 4\alpha_4^{(4)})\alpha_4, \\
O_{16d} &= \alpha_4\gamma_{16} - (\alpha_4^{(2)} + 4\alpha_4^{(4)})\alpha_4^{(2)}, \\
O_{16e} &= \alpha_4\alpha_4^{(4)} - (\alpha_4^{(2)} + 4\alpha_4^{(4)})\gamma_{16}, \\
O_{16f} &= C_4^{(2)}\alpha_4^{(4)} - (\alpha_4^{(2)} + 4\alpha_4^{(4)})^2,
\end{aligned}$$

$$\begin{aligned}
O_{18a} &= u_{18}\gamma_{18} - \alpha_{18}\beta_6, \\
O_{18b} &= u_{18}\epsilon_{18} - \alpha_{18}\delta_{18}, \\
O_{18c} &= \gamma_{18}^2 - u_{18}\beta_6^{(3)}, \\
O_{18d} &= \beta_6\epsilon_{18} - \gamma_{18}\delta_{18}, \\
O_{18e} &= \delta_{18}^2 - \beta_6\beta_6^{(3)}, \\
O_{18f} &= \delta_{18}\epsilon_{18} - \gamma_{18}\beta_6^{(3)}, \\
O_{18A} &= (u_{18} + 3\beta_6)\delta_{18} - \beta_6(\beta_6 + 3\gamma_{18}), \\
O_{18B} &= (\alpha_{18} + 3\gamma_{18})\delta_{18} - \gamma_{18}(\beta_6 + 3\gamma_{18}), \\
O_{18C} &= (\alpha_{18} + 3\gamma_{18})\epsilon_{18} - \beta_6^{(3)}(u_{18} + 3\alpha_{18}), \\
O_{18D} &= \alpha_{18}\gamma_{18} - u_{18}(\delta_{18} + 3\epsilon_{18} - 3\beta_6^{(3)}), \\
O_{18E} &= \gamma_{18}^2 - \beta_6(\delta_{18} + 3\epsilon_{18} - 3\beta_6^{(3)}), \\
O_{18F} &= \alpha_{18}\beta_6^{(3)} - \gamma_{18}(\delta_{18} + 3\epsilon_{18} - 3\beta_6^{(3)}), \\
O_{18G} &= \gamma_{18}\epsilon_{18} - \delta_{18}(\delta_{18} + 3\epsilon_{18} - 3\beta_6^{(3)}), \\
O_{18H} &= \epsilon_{18}^2 - \beta_6^{(3)}(\delta_{18} + 3\epsilon_{18} - 3\beta_6^{(3)}), \\
O_{18I} &= \alpha_{18}^2 - u_{18}\beta_6 - 3\alpha_{18}\gamma_{18} - 6\alpha_{18}\beta_6 + 9\gamma_{18}^2,
\end{aligned}$$

$$\begin{aligned}
O_{25a} &= \alpha_{25}^2 - u_{25}E_{i_5}, \\
O_{25b} &= \alpha_{25}E_{i_5} - u_{25}\gamma_{25}, \\
O_{25c} &= E_{i_5}^2 - \alpha_{25}\gamma_{25}, \\
O_{25d} &= E_{i_5}^2 - u_{25}\delta_{25}, \\
O_{25e} &= E_{i_5}\gamma_{25} - \alpha_{25}\delta_{25}, \\
O_{25f} &= \gamma_{25}^2 - E_{i_5}\delta_{25},
\end{aligned}$$

$$\begin{aligned}
O_{25A} &= u_{25}\iota_{25} - \alpha_{25}(\delta_{25}^2 - 5\beta_5^{(5)}), \\
O_{25B} &= \alpha_{25}\iota_{25} - E_{i_5}(\delta_{25}^2 - 5\beta_5^{(5)}), \\
O_{25C} &= E_{i_5}\iota_{25} - \gamma_{25}(\delta_{25}^2 - 5\beta_5^{(5)}), \\
O_{25D} &= \gamma_{25}\iota_{25} - \delta_{25}(\delta_{25}^2 - 5\beta_5^{(5)}), \\
O_{25E} &= u_{25}\beta_5^{(5)} - \alpha_{25}(\iota_{25} - 2\beta_5^{(5)}), \\
O_{25F} &= \alpha_{25}\beta_5^{(5)} - E_{i_5}(\iota_{25} - 2\beta_5^{(5)}), \\
O_{25G} &= E_{i_5}\beta_5^{(5)} - \gamma_{25}(\iota_{25} - 2\beta_5^{(5)}), \\
O_{25H} &= \gamma_{25}\beta_5^{(5)} - \delta_{25}(\iota_{25} - 2\beta_5^{(5)}), \\
O_{25I} &= (\delta_{25}^2 - 5\beta_5^{(5)})\beta_5^{(5)} - \iota_{25}(\iota_{25} - 2\beta_5^{(5)}).
\end{aligned}$$

5.1. **The case N=8.** We have $O_8 = 0$, since $C_4 = C_4^{(2)} + 8\alpha_4 + 16\alpha_4^{(2)}$ and

$$\begin{aligned}
0 &= E_4^{(2)(2)} - E_4^{(2)(2)} \\
&= (C_2^{(2)2} - 48C_4^{(2)}\alpha_4^{(2)}) - \frac{1}{4}(5C_2^{(2)2} - (C_2^2 - 48C_4\alpha_4)) \\
&= 48O_8.
\end{aligned}$$

5.2. **The case N=16.** We have $O_{16b} \in \mathcal{M}_4(16) \cap \mathbb{C}[[q]]q^9 = \{0\}$. We see

$$\begin{aligned}
C_4\alpha_4^{(2)} &= (C_4^{(2)} + 4\alpha_4)^2/C_4^{(2)} \cdot \alpha_4^2/C_4^{(2)} \\
&= ((C_4^{(2)} + 4\alpha_4)\alpha_4/C_4^{(2)})^2 \\
&= (\alpha_4 + 4\alpha_4^{(2)})^2
\end{aligned}$$

and thus $O_{16f} = (C_4\alpha_4^{(2)} - (\alpha_4 + 4\alpha_4^{(2)})^2)^{(2)} = 0$. Moreover, we have $O_{16c} = \frac{\alpha_4^2}{C_4^{(2)}\gamma_{16} + (\alpha_4^{(2)} + 4\alpha_4^{(4)})\alpha_4} O_{16f} = 0$, $O_{16d} = \frac{\alpha_4}{C_4^{(2)}} O_{16c} = 0$, and $O_{16e} = \frac{\alpha_4^{(4)}}{\gamma_{16}} O_{16d} = 0$.

5.3. **The case N=9.** We get

$$E_4 = C_3^2 + 6^3 C_9 \alpha_9$$

since $E_4 - (C_3^2 + 6^3 C_9 \alpha_9) \in \mathcal{M}_4(9) \cap \mathbb{C}[[q]]q^5 = \{0\}$. We note $E_4^{(3)} = C_3^2 - 24C_9\alpha_9$ and thus $\alpha_3 = C_9\alpha_9$. In $\mathbb{C}[C_3, C_9, \alpha_9, E_6]$, we get

$$\begin{aligned}
O_3 &= \frac{1}{6^3}(27C_3^4 - 18C_3^2E_4 - E_4^2 - 8C_3E_6) \\
&= \frac{1}{3^3}(C_3^4 - 540C_3^2C_9\alpha_9 - 5832C_9^2\alpha_9^2 - C_3E_6), \\
O_3^{(3)} &= \frac{1}{3^7}(C_3^4 - 4C_3^3C_9 - 540C_3^2C_9\alpha_9 - 864C_3C_9^3 + 2160C_3C_9^2\alpha_9 + 864C_9^4 \\
&\quad + 7776C_9^3\alpha_9 - 5832C_9^2\alpha_9^2 - C_3E_6 + 4C_9E_6),
\end{aligned}$$

thus

$$\begin{aligned}
0 &= \frac{27}{4}(O_3 - 3^4 O_3^{(3)})/C_9 \\
&= C_3^3 + 216C_3C_9^2 - 540C_3C_9\alpha_9 - 216C_9^3 - 1944C_9^2\alpha_9 - E_6.
\end{aligned}$$

Hence, we see

$$\begin{aligned}\beta_3 &= \frac{1}{108^2}(7C_3^3 - 5C_3E_4 - 2E_6) \\ &= \frac{1}{27}C_9^2(-C_3 + C_9 + 9\alpha_9) \\ &= C_9^2\beta_9\end{aligned}$$

and

$$C_3\beta_9 = C_3\beta_3/C_9^2 = \alpha_3^2/C_9^2 = (C_9\alpha_9)^2/C_9^2 = \alpha_9,$$

i.e. $O_9 = 0$.

5.4. The case $N=18$. We put

$$\begin{aligned}O_{18B'} &= (u_{18} + 3\beta_6)\epsilon_{18} - \gamma_{18}(\beta_6 + 3\gamma_{18}), \\ O_{18X} &= \alpha_{18}\gamma_{18} - \beta_6^2 - 3\gamma_{18}^2 - 6\beta_6\gamma_{18} + 9\beta_6\beta_6^{(3)},\end{aligned}$$

then we see

$$\begin{aligned}O_{18B'} &= O_{18B} + O_{18b} + 3O_{18d}, \\ O_{18X} &= 3O_{18c} + O_{18A} + 3O_{18B'} + O_{18D} + 3O_{18E}.\end{aligned}$$

First have

$$O_{18c} = O_6^{(3)} = 0.$$

We see

$$\begin{aligned}0 &= O_6 \\ &= -u_{18}\beta_6 + \alpha_{18}^2 - 6\alpha_{18}\beta_6 - 3\beta_6^2 - 18\beta_6\gamma_{18} + 27\beta_6\beta_6^{(3)} \\ &= O_{18I} + 3O_{18X},\end{aligned}$$

$$\begin{aligned}0 &= E_4^{(3)} - E_4^{(3)} \\ &= (C_2^2 + C_2C_3 - C_3^2 + 5C_3C_6 - C_6^2)^{(3)} - (C_3^2 - 24C_9\alpha_9) \\ &= 16(-u_{18}\beta_6 + 12u_{18}\gamma_{18} + 9u_{18}\beta_6^{(3)} + \alpha_{18}^2 - 18\alpha_{18}\beta_6 - 3\beta_6^2 - 18\beta_6\gamma_{18} + 27\beta_6\beta_6^{(3)} - 9\gamma_{18}^2) \\ &= 16(12O_{18a} - 9O_{18c} + O_{18I} + 3O_{18X}),\end{aligned}$$

thus $O_{18a} = 0$. We see $O_{18X} = \frac{\gamma_{18}}{\alpha_{18}}O_{18I}$ thus $O_I = O_X = 0$. Next, we also see $O_{18C} = \frac{\alpha_{18}}{u_{18}}O_{18B'}$, $O_{18B} = \frac{\alpha_{18}}{u_{18}}O_{18A}$ and

$$\begin{aligned}0 &= E_4 - E_4 \\ &= (C_2^2 + C_2C_3 - C_3^2 + 5C_3C_6 - C_6^2) - (C_3^2 + 6^3C_9\alpha_9) \\ &= 72(-11u_{18}\beta_6 - 21u_{18}\gamma_{18} + 27u_{18}\delta_{18} + 9u_{18}\epsilon_{18} + 11\alpha_{18}^2 - 45\alpha_{18}\beta_6 - 36\alpha_{18}\gamma_{18} \\ &\quad + 81\alpha_{18}\delta_{18} + 27\alpha_{18}\epsilon_{18} - 81\alpha_{18}\beta_6^{(3)} - 24\beta_6^2 - 153\beta_6\gamma_{18} + 81\beta_6\delta_{18} + 27\beta_6\epsilon_{18} \\ &\quad - 27\beta_6\beta_6^{(3)} - 189\gamma_{18}^2 + 243\gamma_{18}\delta_{18} + 81\gamma_{18}\epsilon_{18}) \\ &= 72(-21O_{18a} - 36O_{18c} + 24O_{18A} + 81O_{18B} + 27O_{18C} - 3O_{18D} - 9O_{18E} + 11O_{18I}) \\ &= 72 \cdot 3(8O_{18A} + 27O_{18B} + 9O_{18C} - O_{18D} - 3O_{18E}) \\ &= 72 \cdot 3 \cdot 3 \cdot (3\frac{\alpha_{18}}{u_{18}} + 1)(3O_{18A} + O_{18B'}),\end{aligned}$$

$$\begin{aligned}
0 &= E_4^{(2)} - E_4^{(2)} \\
&= \frac{1}{4}(5C_2^2 - (C_2^2 + C_2C_3 - C_3^2 + 5C_3C_6 - C_6^2)) - (C_3^2 + 6^3C_9\alpha_9)^{(2)} \\
&= 108(-3u_{18}\beta_6 - 4u_{18}\gamma_{18} - 6u_{18}\delta_{18} - 6u_{18}\epsilon_{18} + 18u_{18}\beta_6^{(3)} + 3\alpha_{18}^2 - 14\alpha_{18}\beta_6 \\
&\quad - 3\beta_6^2 - 30\beta_6\gamma_{18} - 18\beta_6\delta_{18} - 18\beta_6\epsilon_{18} + 81\beta_6\beta_6^{(3)}) \\
&= 18(-4O_{18a} + 9O_{18c} + 3O_{18A} + 21O_{18B'} + 9O_{18D} + 27O_{18E} + 3O_{18I}) \\
&= 18 \cdot 3(O_{18A} + 7O_{18B'} + 3O_{18D} + 9O_{18E}) \\
&= -18 \cdot 3 \cdot 2(O_{18A} + O_{18B'}),
\end{aligned}$$

thus $O_{18A} = O_{18B'} = 0$ and $O_{18B} = O_{18C} = 0$. We see $O_{18d} = \frac{\gamma_{18}}{\alpha_{18}}O_{18b}$ and

$$O_{18b} + 3O_{18d} = O_{18B'} - O_{18B} = 0,$$

thus $O_{18b} = O_{18d} = 0$. Moreover, we see $O_{18E} = \frac{\beta_6}{u_{18}}O_{18D}$ and

$$O_{18D} + 3O_{18E} = O_{18X} = 0,$$

thus $O_{18D} = O_{18E} = 0$, $O_{18F} = \frac{\gamma_{18}}{u_{18}}O_{18D} = 0$, $O_{18G} = \frac{\epsilon_{18}}{\gamma_{18}}O_{18E} = 0$. Finally we see

$$O_{18H} = \frac{\beta_6^{(3)}}{\delta_{18}}O_{18G} + \frac{\epsilon_{18}}{\delta_{18}}O_{18f} = \frac{\epsilon_{18}^2}{\delta_{18}^2}O_{18e},$$

$$\begin{aligned}
0 &= O_9^{(2)} \\
&= -u_{18}\delta_{18} - u_{18}\epsilon_{18} + 2u_{18}\beta_6^{(3)} + \beta_6^2 + 4\beta_6\gamma_{18} - 6\beta_6\delta_{18} - 6\beta_6\epsilon_{18} + 6\beta_6\beta_6^{(3)} + 4\gamma_{18}^2 \\
&\quad - 6\gamma_{18}\delta_{18} - 6\gamma_{18}\epsilon_{18} + 9\delta_{18}^2 + 18\delta_{18}\epsilon_{18} - 27\delta_{18}\beta_6^{(3)} + 9\epsilon_{18}^2 - 27\epsilon_{18}\beta_6^{(3)} + 27\beta_6^{(3)2} \\
&= -O_{18b} - 2O_{18c} + 3O_{18d} + 3O_{18e} - O_{18A} - O_{18B} + 3O_{18E} - 6O_{18G} + 9O_{18H} \\
&= 3(1 + 3\frac{\epsilon_{18}^2}{\delta_{18}^2})O_{18e},
\end{aligned}$$

thus $O_{18e} = O_{18f} = O_{18H} = 0$.

5.5. The case N=5,10. First, in $\mathcal{M}(5)$ we get

$$E_6 = \frac{1}{8}C_5(2000C_5^2 - 117E_4 - 1875E_4^{(5)}).$$

We have $O_{10a} \in \mathcal{M}_4(10) \cap \mathbb{C}[[q]]q^7 = \{0\}$. We see $O_{10c} = \frac{\alpha_{10}}{u_{10}}O_{10b}$,

$$O_{10e} = \frac{\alpha_{10}}{u_{10}}O_{10d} - 20\frac{\beta_{10}}{u_{10}}O_{10b},$$

and

$$\begin{aligned}
0 &= E_6 - E_6 \\
&= C_2(4C_2^2 - 3E_4) - \frac{1}{8}C_5(2000C_5^2 - 117E_4 - 1875E_4^{(5)}) \\
&= 12(-1595u_{10}^2\beta_{10} + 1595u_{10}\alpha_{10}^2 - 14406u_{10}\alpha_{10}\beta_{10} - 50612u_{10}\beta_{10}^2 + 14406\alpha_{10}^3 \\
&\quad + 50612\alpha_{10}^2\beta_{10} - 216\alpha_{10}\beta_{10}^2 - 29520\beta_{10}^3 + 216u_{10}\epsilon_{10} + 12240u_{10}\zeta_{10} + 19008\alpha_{10}\epsilon_{10} \\
&\quad + 177120\alpha_{10}\zeta_{10} + 63360\beta_{10}\epsilon_{10} + 590400\beta_{10}\zeta_{10}) \\
&= 12((1595u_{10} + 14406\alpha_{10} + 50612\beta_{10})O_{10a} - 72(170O_{10b} + 2460O_{10c} + 3O_{10d} + 410O_{10e})) \\
&= 12 \cdot 72((170 + 2460\frac{\alpha_{10}}{u_{10}} - 8200\frac{\beta_{10}}{u_{10}})O_{10b} - (3 + 410\frac{\alpha_{10}}{u_{10}})O_{10d}),
\end{aligned}$$

$$\begin{aligned}
0 &= E_6^{(2)} - E_6^{(2)} \\
&= \frac{1}{8}C_2(11C_2^2 - 3E_4) - \frac{1}{8}C_5^{(2)}(2000C_5^{(2)2} - \frac{117}{4}(5C_2^2 - E_4) - \frac{1875}{4}(5C_2^2 - E_4)^{(5)}) \\
&= 3(455u_{10}^2\beta_{10} - 455u_{10}\alpha_{10}^2 + 5058u_{10}\alpha_{10}\beta_{10} + 16736u_{10}\beta_{10}^2 - 5058\alpha_{10}^3 - 16736\alpha_{10}^2\beta_{10} \\
&\quad - 1152\alpha_{10}\beta_{10}^2 + 1080\beta_{10}^3 + 1152u_{10}\epsilon_{10} - 9720u_{10}\zeta_{10} + 17856\alpha_{10}\epsilon_{10} - 38160\alpha_{10}\zeta_{10} \\
&\quad + 52560\beta_{10}\epsilon_{10} - 21600\beta_{10}\zeta_{10}) \\
&= 3(- (455u_{10} + 5058\alpha_{10} + 16736\beta_{10})O_{10a} + 72(135O_{10b} + 530O_{10c} - 16O_{10d} + 15O_{10e})) \\
&= 3 \cdot 72((135 + 530\frac{\alpha_{10}}{u_{10}} - 300\frac{\beta_{10}}{u_{10}})O_{10b} - (16 - 15\frac{\alpha_{10}}{u_{10}})O_{10d}).
\end{aligned}$$

Thus we have $O_{10b} = O_{10c} = O_{10d} = O_{10e} = 0$, and $O_{10f} = \frac{\zeta_{10}}{\beta_{10}}O_{10e} = 0$.

We put

$$O_{10e'} = \beta_{10}^3 - (u_{10} + 8\alpha_{10} + 20\beta_{10})\zeta_{10},$$

then we see $O_{10e'} = O_{10e} + O_{10b} + 8O_{10c} = 0$. We get

$$\begin{aligned}
E_6^{(5)} &= (C_2(4C_2^2 - 3E_4))^{(5)} \\
&= 12(13u_{10}^2\beta_{10} - 13u_{10}\alpha_{10}^2 + 162u_{10}\alpha_{10}\beta_{10} + 556u_{10}\beta_{10}^2 - 162\alpha_{10}^3 - 556\alpha_{10}^2\beta_{10} + 72\alpha_{10}\beta_{10}^2 \\
&\quad + 432\beta_{10}^3 - 72u_{10}\epsilon_{10} - 432u_{10}\zeta_{10} - 576\alpha_{10}\epsilon_{10} - 3744\alpha_{10}\zeta_{10} - 1152\beta_{10}\epsilon_{10} - 8640\beta_{10}\zeta_{10}) \\
&= u_{10}^3 + 18u_{10}^2\alpha_{10} - 120u_{10}^2\beta_{10} + 240u_{10}\alpha_{10}^2 - 1656u_{10}\alpha_{10}\beta_{10} - 6528u_{10}\beta_{10}^2 \\
&\quad + 2016\alpha_{10}^3 + 7008\alpha_{10}^2\beta_{10} - 576\alpha_{10}\beta_{10}^2 - 5120\beta_{10}^3 + 288u_{10}\epsilon_{10} + 2304u_{10}\zeta_{10} \\
&\quad + 3456\alpha_{10}\epsilon_{10} + 27648\alpha_{10}\zeta_{10} + 11520\beta_{10}\epsilon_{10} + 92160\beta_{10}\zeta_{10} \\
&\quad + 12(-(162\alpha_{10} + 556\beta_{10} + 13u_{10})O_{10a} + 72(4O_{10c} + O_{10d} + 6O_{10e'})) \\
&= \frac{1}{40}C_5(-80C_5^2 + 3E_4 + 117E_4^{(5)}).
\end{aligned}$$

At last, we have

$$O_5 = \frac{1}{124.5}(3520C_5^4 + E_4^2 + 625E_4^{(5)2} - 160C_5^2E_4 - 4000C_5^2E_4^{(5)} + 14E_4E_4^{(5)})$$

in $\mathbb{C}[C_5, E_4, E_4^{(5)}]$ and

$$\begin{aligned}
O_5 &= \frac{4}{81}(45u_{10}^3\beta_{10} - 45u_{10}^2\alpha_{10}^2 + 1098u_{10}^2\alpha_{10}\beta_{10} + 4157u_{10}^2\beta_{10}^2 - 1098u_{10}\alpha_{10}^3 \\
&\quad + 1694u_{10}\alpha_{10}^2\beta_{10} + 42480u_{10}\alpha_{10}\beta_{10}^2 + 74160u_{10}\beta_{10}^3 - 5851\alpha_{10}^4 - 42480\alpha_{10}^3\beta_{10} \\
&\quad - 73188\alpha_{10}^2\beta_{10}^2 + 12960\alpha_{10}\beta_{10}^3 + 42768\beta_{10}^4 - 324u_{10}^2\zeta_{10} - 648u_{10}\alpha_{10}\epsilon_{10} \\
&\quad - 10368u_{10}\alpha_{10}\zeta_{10} - 2880u_{10}\beta_{10}\epsilon_{10} - 43200u_{10}\beta_{10}\zeta_{10} - 7488\alpha_{10}^2\epsilon_{10} - 68256\alpha_{10}^2\zeta_{10} \\
&\quad - 54432\alpha_{10}\beta_{10}\epsilon_{10} - 508032\alpha_{10}\beta_{10}\zeta_{10} - 93312\beta_{10}^2\epsilon_{10} - 860544\beta_{10}^2\zeta_{10} + 5184\epsilon_{10}^2 \\
&\quad + 41472\epsilon_{10}\zeta_{10} + 103680\zeta_{10}^2) \\
&= \frac{4}{81}((-45u_{10}^2 - 1098u_{10}\alpha_{10} - 4157u_{10}\beta_{10} - 5851\alpha_{10}^2 - 42480\alpha_{10}\beta_{10} - 74160\beta_{10}^2 \\
&\quad + 2880\epsilon_{10} + 432\zeta_{10})O_{10a} + 324((u_{10} - 8\alpha_{10})O_{10b} - 36(3\alpha_{10} + 8\beta_{10})O_{10c} + 3\alpha_{10}O_{10d} \\
&\quad + 4(10\alpha_{10} + 33\beta_{10})O_{10e'} - 16O_{10f})) \\
&= 0.
\end{aligned}$$

5.6. The case $N=25$. We have $O_{25d}, O_{25e}, O_{25f} \in \mathcal{M}_4(25) \cap \mathbb{C}[[q]]q^{11} = \{0\}$, thus we have also

$$O_{25c} = \frac{E_{i5}}{\gamma_{25}}O_{25e} = 0, \quad O_{25b} = \frac{\alpha_{25}}{E_{i5}}O_{25d} = 0, \quad O_{25a} = \frac{\alpha_{25}}{E_{i5}}O_{25b} = 0.$$

We have $O_{25D}, O_{25H} \in \mathcal{M}_6(25) \cap \mathbb{C}[[q]]q^{15} = \{0\}$, thus

$$\begin{aligned}
O_{25C} &= \frac{E_{i5}}{\gamma_{25}}O_{25D} = 0, \quad O_{25B} = \frac{\alpha_{25}}{E_{i5}}O_{25C} = 0, \quad O_{25A} = \frac{u_{25}}{\alpha_{25}}O_{25B} = 0, \\
O_{25G} &= \frac{E_{i5}}{\gamma_{25}}O_{25H} = 0, \quad O_{25F} = \frac{\alpha_{25}}{E_{i5}}O_{25G} = 0, \quad O_{25E} = \frac{u_{25}}{\alpha_{25}}O_{25F} = 0,
\end{aligned}$$

$$O_{25I} = \frac{\iota_{25}}{\delta_{25}} O_{25H} = 0.$$

Moreover we get

$$\begin{aligned} \frac{1}{5}C_5(-80C_5^2 + 3E_4 + 117E_4^{(5)}) &= 8E_6^{(5)} \\ &= C_5^{(5)}(2000C_5^{(5)2} - 117E_4^{(5)} - 1875E_4^{(25)}), \end{aligned}$$

thus

$$\begin{aligned} E_4^{(25)} &= \frac{1}{1875C_5^{(5)}}(C_5^{(5)}(2000C_5^{(5)2} - 117E_4^{(5)}) - \frac{1}{5}C_5(-80C_5^2 + 3E_4 + 117E_4^{(5)})) \\ &= \frac{1}{625C_5^{(5)}}(625u_{25}^3 + 7500u_{25}^2\alpha_{25} + 18009u_{25}^2E_{i_5} + 20410u_{25}^2\gamma_{25} + 10805u_{25}^2\delta_{25} \\ &\quad + 28866u_{25}\alpha_{25}^2 + 141044u_{25}\alpha_{25}E_{i_5} + 177680u_{25}\alpha_{25}\gamma_{25} + 104440u_{25}\alpha_{25}\delta_{25} \\ &\quad + 165051u_{25}E_{i_5}^2 + 425488u_{25}E_{i_5}\gamma_{25} + 360000u_{25}E_{i_5}\delta_{25} + 212794u_{25}\gamma_{25}^2 \\ &\quad + 427972u_{25}\gamma_{25}\delta_{25} + 198961u_{25}\delta_{25}^2 + 32296\alpha_{25}^3 + 227714\alpha_{25}^2E_{i_5} + 362740\alpha_{25}^2\gamma_{25} \\ &\quad + 223370\alpha_{25}^2\delta_{25} + 419832\alpha_{25}E_{i_5}^2 + 1571832\alpha_{25}E_{i_5}\gamma_{25} + 1762060\alpha_{25}E_{i_5}\delta_{25} \\ &\quad + 1132456\alpha_{25}\gamma_{25}^2 + 3316528\alpha_{25}\gamma_{25}\delta_{25} + 1802164\alpha_{25}\delta_{25}^2 - 108621E_{i_5}^3 - 378738E_{i_5}^2\gamma_{25} \\ &\quad + 1991655E_{i_5}^2\delta_{25} - 2675894E_{i_5}\gamma_{25}^2 + 5762228E_{i_5}\gamma_{25}\delta_{25} + 7828569E_{i_5}\delta_{25}^2 \\ &\quad - 5652860\gamma_{25}^3 - 6961110\gamma_{25}^2\delta_{25} + 317070\gamma_{25}\delta_{25}^2 + 121205\delta_{25}^3 - 11532u_{25}\iota_{25} \\ &\quad - 8616u_{25}\beta_5^{(5)} - 61968\alpha_{25}\iota_{25} - 24384\alpha_{25}\beta_5^{(5)} - 198828E_{i_5}\iota_{25} - 17064E_{i_5}\beta_5^{(5)} \\ &\quad - 352920\gamma_{25}\iota_{25} + 65040\gamma_{25}\beta_5^{(5)} - 374460\delta_{25}\iota_{25} + 158520\delta_{25}\beta_5^{(5)}) \\ &= \frac{1}{625C_5^{(5)}}(625u_{25}^3 + 7500u_{25}^2\alpha_{25} + 18585u_{25}^2E_{i_5} + 22650u_{25}^2\gamma_{25} + 14325u_{25}^2\delta_{25} \\ &\quad + 28290u_{25}\alpha_{25}^2 + 137940u_{25}\alpha_{25}E_{i_5} + 217200u_{25}\alpha_{25}\gamma_{25} + 151800u_{25}\alpha_{25}\delta_{25} \\ &\quad + 110715u_{25}E_{i_5}^2 + 325200u_{25}E_{i_5}\gamma_{25} + 489600u_{25}E_{i_5}\delta_{25} - 63750u_{25}\gamma_{25}^2 \\ &\quad + 226500u_{25}\gamma_{25}\delta_{25} + 56625u_{25}\delta_{25}^2 + 33160\alpha_{25}^3 + 239010\alpha_{25}^2E_{i_5} + 489300\alpha_{25}^2\gamma_{25} \\ &\quad + 369450\alpha_{25}^2\delta_{25} + 346200\alpha_{25}E_{i_5}^2 + 1517400\alpha_{25}E_{i_5}\gamma_{25} + 2239500\alpha_{25}E_{i_5}\delta_{25} \\ &\quad + 105000\alpha_{25}\gamma_{25}^2 + 1458000\alpha_{25}\gamma_{25}\delta_{25} + 412500\alpha_{25}\delta_{25}^2 - 53325E_{i_5}^3 + 372750E_{i_5}^2\gamma_{25} \\ &\quad + 2886375E_{i_5}^2\delta_{25} - 1569750E_{i_5}\gamma_{25}^2 + 3772500E_{i_5}\gamma_{25}\delta_{25} + 1625625E_{i_5}\delta_{25}^2 \\ &\quad - 2277500\gamma_{25}^3 - 813750\gamma_{25}^2\delta_{25} + 168750\gamma_{25}\delta_{25}^2 + 3125\delta_{25}^3 - 7500u_{25}\iota_{25} \\ &\quad + 15000u_{25}\beta_5^{(5)} - 30000\alpha_{25}\iota_{25} + 60000\alpha_{25}\beta_5^{(5)} - 67500E_{i_5}\iota_{25} + 135000E_{i_5}\beta_5^{(5)} \\ &\quad - 75000\gamma_{25}\iota_{25} + 150000\gamma_{25}\beta_5^{(5)} - 37500\delta_{25}\iota_{25} + 75000\delta_{25}\beta_5^{(5)}) \\ &\quad + 32((18u_{25} - 27\alpha_{25} - 1588E_{i_5} - 3955\gamma_{25} - 4565\delta_{25})O_{25a} \\ &\quad + (70u_{25} + 1235\alpha_{25} + 821E_{i_5})O_{25b} - 1701E_{i_5}O_{25c} \\ &\quad + (110u_{25} + 1480\alpha_{25} - 27E_{i_5} - 6296\gamma_{25} - 4448\delta_{25})O_{25d} \\ &\quad - (17188E_{i_5} + 58079\gamma_{25} + 43301\delta_{25})O_{25e} \\ &\quad + (8642u_{25} + 32108\alpha_{25} + 23512E_{i_5} - 105480\gamma_{25} - 192105\delta_{25})O_{25f} \\ &\quad - 126O_{25A} - 1737O_{25B} - 4635O_{25C} - 3690O_{25D} - 738O_{25E} - 531O_{25F} \\ &\quad + 4995O_{25G} + 10530O_{25H})) \\ &= \frac{1}{625}(52(C_5 - 3C_{25})^2 + 432C_{25}^2 - 60E_{\rho_5}^2 + 600E_{r_5}^2 + 600E_{i_5}^2 - E_4 - 14E_4^{(5)}). \end{aligned}$$

5.7. The case N=7. We have $O_{7a}, O_{7b}, O_{7d} \in \mathcal{M}_8(7) \cap \mathbb{C}[[q]]q^5 = \{0\}$, thus we have also

$$O_{7c} = \frac{\beta_7}{C_7} O_{7b} = 0, \quad O_{7e} = \frac{\beta_7}{C_7} O_{7d} = 0, \quad O_{7f} = \frac{\beta_7}{C_7} O_{7e} = 0.$$

6. DECOMPOSITION OF POLYNOMIAL RINGS

Suppose that R is a ring. If $O \in R + RY + Y^2$, then by induction on m , we get $RY^m \subset (O) + R + RY$, thus

$$R[Y] = (O) \oplus R \oplus RY.$$

By virtue of this, for example, we see $\mathbb{C}[C_3^{(2)}, \alpha_6, \beta_6] = (O_6) \oplus \mathbb{C}[C_3^{(2)}, \beta_6] \oplus \mathbb{C}[C_3^{(2)}, \beta_6]\alpha_6$. Next, if $O \in XZ + R[Z]$, then we see $R[Z]X \subset (R + R[Z]Z)X \subset RX + (O) + R[Z]$ and

$$R[X, Z] = (O) + R[X] + R[Z].$$

For example, we see $\mathbb{C}[C_3^{(2)}, \alpha_6, \beta_6] = (O_6) + \mathbb{C}[\alpha_6][C_3^{(2)}] + \mathbb{C}[\alpha_6][\beta_6]$. If $O \in XZ + R[Z, Z']$ and $O' \in XZ' + R[Z, Z']$, then we get

$$R[X, Z, Z'] = (O, O') + R[X] + R[Z, Z']$$

in a similar fashion. If $O \in XZ + R[Z, Z', Z'']$, $O' \in XZ' + R[Z, Z', Z'']$ and $O'' \in XZ'' + R[Z, Z', Z'']$, then we get

$$R[X, Z, Z', Z''] = (O, O', O'') + R[X] + R[Z, Z', Z''].$$

6.1. The case $N=12, 16$. Put $R_{12} = \mathbb{C}[C_3^{(2)}, \alpha_6, \beta_6, \gamma_{12}, \beta_6^{(2)}]$ and

$$I_{12} = (O_6, O_{12b}, O_{12c}, O_{12d}, O_{12e}, O_{12f}).$$

We see

$$\begin{aligned} \mathbb{C}[\beta_6, \gamma_{12}, \beta_6^{(2)}] &= (O_{12b}) + \mathbb{C}[\gamma_{12}][\beta_6] + \mathbb{C}[\gamma_{12}][\beta_6^{(2)}], \\ \mathbb{C}[\alpha_6, \beta_6, \gamma_{12}, \beta_6^{(2)}] &= (O_{12d}, O_{12e}) + \mathbb{C}[\beta_6][\alpha_6] + \mathbb{C}[\beta_6][\gamma_{12}, \beta_6^{(2)}], \end{aligned}$$

and

$$\begin{aligned} R_{12} &= (O_{12c}, O_{12f}) + \mathbb{C}[\alpha_6, \beta_6][C_3^{(2)}] + \mathbb{C}[\alpha_6, \beta_6][\gamma_{12}, \beta_6^{(2)}] \\ &= I_{12} + \mathbb{C}[C_3^{(2)}, \alpha_6] + \mathbb{C}[\alpha_6, \beta_6] + \mathbb{C}[\beta_6, \gamma_{12}] + \mathbb{C}[\gamma_{12}, \beta_6^{(2)}]. \end{aligned}$$

Put $R_{16} = \mathbb{C}[C_4^{(2)}, \alpha_4, \alpha_4^{(2)}, \gamma_{16}, \alpha_4^{(4)}]$ and

$$I_{16} = (O_8, O_{16b}, O_{16c}, O_{16d}, O_{16e}, O_{16f}).$$

Similarly, we see

$$R_{16} = I_{16} + \mathbb{C}[C_4^{(2)}, \alpha_4] + \mathbb{C}[\alpha_4, \alpha_4^{(2)}] + \mathbb{C}[\alpha_4^{(2)}, \gamma_{16}] + \mathbb{C}[\gamma_{16}, \alpha_4^{(4)}].$$

6.2. The case $N=18$. Put $R_{18} = \mathbb{C}[u_{18}, \alpha_{18}, \beta_6, \gamma_{18}, \delta_{18}, \epsilon_{18}, \beta_6^{(3)}]$ and

$$I_{18} = (O_{18a}, O_{18b}, O_{18c}, O_{18d}, O_{18e}, O_{18f}, O_{18A}, O_{18B}, O_{18C}, O_{18D}, O_{18E}, O_{18F}, O_{18G}, O_{18H}, O_{18I}).$$

We see

$$\begin{aligned} \mathbb{C}[\delta_{18}, \epsilon_{18}, \beta_6^{(3)}] &= (O_{18H}) + \mathbb{C}[\epsilon_{18}][\delta_{18}] + \mathbb{C}[\epsilon_{18}][\beta_6^{(3)}], \\ \mathbb{C}[\gamma_{18}, \delta_{18}, \epsilon_{18}, \beta_6^{(3)}] &= (O_{18G}, O_{18f}) + \mathbb{C}[\delta_{18}][\gamma_{18}] + \mathbb{C}[\delta_{18}][\epsilon_{18}, \beta_6^{(3)}], \end{aligned}$$

and

$$\begin{aligned} \mathbb{C}[\beta_6, \gamma_{18}, \delta_{18}, \epsilon_{18}, \beta_6^{(3)}] &= (O_{18E}, O_{18d}, O_{18e}) + \mathbb{C}[\gamma_{18}][\beta_6] + \mathbb{C}[\gamma_{18}][\delta_{18}, \epsilon_{18}, \beta_6^{(3)}] \\ &\subset I_{18} + \mathbb{C}[\beta_6, \gamma_{18}] + \mathbb{C}[\gamma_{18}, \delta_{18}] + \mathbb{C}[\delta_{18}, \epsilon_{18}] + \mathbb{C}[\epsilon_{18}, \beta_6^{(3)}]. \end{aligned}$$

Moreover, we see

$$\mathbb{C}[\alpha_{18}, \beta_6, \gamma_{18}, \delta_{18}] = (O_{18X} + 9O_{18e}, O_{18B}) + \mathbb{C}[\beta_6][\alpha_{18}] + \mathbb{C}[\beta_6][\gamma_{18}, \delta_{18}],$$

$$\begin{aligned}\mathbb{C}[\alpha_{18}, \beta_6, \gamma_{18}, \delta_{18}, \epsilon_{18}, \beta_6^{(3)}] &= (O_{18C} - O_{18c}, O_{18F}) + \mathbb{C}[\beta_6, \gamma_{18}, \delta_{18}][\alpha_{18}] \\ &\quad + \mathbb{C}[\beta_6, \gamma_{18}, \delta_{18}][\epsilon_{18}, \beta_6^{(3)}],\end{aligned}$$

$$\mathbb{C}[u_{18}, \alpha_{18}, \beta_6, \gamma_{18}] = (O_{18I}, O_{18a}) + \mathbb{C}[\alpha_{18}][u_{18}] + \mathbb{C}[\alpha_{18}][\beta_6, \gamma_{18}],$$

and

$$\begin{aligned}R_{18} &= (O_{18A}, O_{18b}, O_{18c}) + \mathbb{C}[\alpha_{18}, \beta_6, \gamma_{18}][u_{18}] + \mathbb{C}[\alpha_{18}, \beta_6, \gamma_{18}][\delta_{18}, \epsilon_{18}, \beta_6^{(3)}] \\ &= I_{18} + \mathbb{C}[u_{18}, \alpha_{18}] + \mathbb{C}[\alpha_{18}, \beta_6] + \mathbb{C}[\beta_6, \gamma_{18}, \delta_{18}, \epsilon_{18}, \beta_6^{(3)}].\end{aligned}$$

6.3. The case N=10. Put $R_{10} = \mathbb{C}[u_{10}, \alpha_{10}, \beta_{10}, \epsilon_{10}, \zeta_{10}]$ and

$$I_{10} = (O_{10a}, O_{10b}, O_{10c}, O_{10d}, O_{10e}, O_{10f}).$$

We see

$$\begin{aligned}\mathbb{C}[u_{10}, \alpha_{10}, \beta_{10}] &= (O_{10a}) + \mathbb{C}[\alpha_{10}][u_{10}] + \mathbb{C}[\alpha_{10}][\beta_{10}], \\ \mathbb{C}[\beta_{10}, \epsilon_{10}, \zeta_{10}] &= (O_{10f}) \oplus \mathbb{C}[\beta_{10}, \zeta_{10}] \oplus \mathbb{C}[\beta_{10}, \zeta_{10}]\epsilon_{10}, \\ \mathbb{C}[\alpha_{10}, \beta_{10}, \epsilon_{10}, \zeta_{10}] &= (O_{10e}, O_{10c}) + \mathbb{C}[\beta_{10}][\alpha_{10}] + \mathbb{C}[\beta_{10}][\epsilon_{10}, \zeta_{10}],\end{aligned}$$

and

$$\begin{aligned}R_{10} &= (O_{10d}, O_{10b}) + \mathbb{C}[\alpha_{10}, \beta_{10}][u_{10}] + \mathbb{C}[\alpha_{10}, \beta_{10}][\epsilon_{10}\zeta_{10}] \\ &= I_{10} + \mathbb{C}[u_{10}, \alpha_{10}] + \mathbb{C}[\alpha_{10}, \beta_{10}] + \mathbb{C}[\beta_{10}, \zeta_{10}] + \mathbb{C}[\beta_{10}, \zeta_{10}]\epsilon_{10}.\end{aligned}$$

6.4. The case N=25. Put $R_{25} = \mathbb{C}[u_{25}, \alpha_{25}, E_{i_5}, \gamma_{25}, \delta_{25}, \iota_{25}, \beta_5^{(5)}]$ and

$$I_{25} = (O_{25a}, O_{25b}, O_{25c}, O_{25d}, O_{25e}, O_{25f}, O_{25A}, O_{25B}, O_{25C}, O_{25D}, O_{25E}, O_{25F}, O_{25G}, O_{25H}, O_{25I}).$$

We see

$$\begin{aligned}\mathbb{C}[u_{25}, \alpha_{25}, E_{i_5}] &= (O_{25a}) + \mathbb{C}[\alpha_{25}][u_{25}] + \mathbb{C}[\alpha_{25}][E_{i_5}], \\ \mathbb{C}[E_{i_5}, \gamma_{25}, \delta_{25}] &= (O_{25f}) + \mathbb{C}[\gamma_{25}][E_{i_5}] + \mathbb{C}[\gamma_{25}][\delta_{25}], \\ \mathbb{C}[\alpha_{25}, E_{i_5}, \gamma_{25}, \delta_{25}] &= (O_{25c}, O_{25e}) + \mathbb{C}[E_{i_5}][\alpha_{25}] + \mathbb{C}[E_{i_5}][\gamma_{25}, \delta_{25}],\end{aligned}$$

and

$$\begin{aligned}\mathbb{C}[u_{25}, \alpha_{25}, E_{i_5}, \gamma_{25}, \delta_{25}] &= (O_{25b}, O_{25d}) + \mathbb{C}[\alpha_{25}, E_{i_5}][u_{25}] + \mathbb{C}[\alpha_{25}, E_{i_5}][\gamma_{25}, \delta_{25}] \\ &\subset I_{25} + \mathbb{C}[u_{25}, \alpha_{25}] + \mathbb{C}[\alpha_{25}, E_{i_5}] + \mathbb{C}[E_{i_5}, \gamma_{25}] + \mathbb{C}[\gamma_{25}, \delta_{25}].\end{aligned}$$

Moreover, we see

$$\begin{aligned}\mathbb{C}[\delta_{25}, \iota_{25}, \beta_5^{(5)}] &= (O_{25I}) \oplus \mathbb{C}[\delta_{25}, \beta_5^{(5)}] \oplus \mathbb{C}[\delta_{25}, \beta_5^{(5)}]\iota_{25}, \\ \mathbb{C}[\gamma_{25}, \delta_{25}, \iota_{25}, \beta_5^{(5)}] &= (O_{25D}, O_{25H}) + \mathbb{C}[\delta_{25}][\gamma_{25}] + \mathbb{C}[\delta_{25}][\iota_{25}, \beta_5^{(5)}], \\ \mathbb{C}[E_{i_5}, \gamma_{25}, \delta_{25}, \iota_{25}, \beta_5^{(5)}] &= (O_{25C}, O_{25G}) + \mathbb{C}[\gamma_{25}, \delta_{25}][E_{i_5}] + \mathbb{C}[\gamma_{25}, \delta_{25}][\iota_{25}, \beta_5^{(5)}], \\ \mathbb{C}[\alpha_{25}, E_{i_5}, \gamma_{25}, \delta_{25}, \iota_{25}, \beta_5^{(5)}] &= (O_{25B}, O_{25F}) + \mathbb{C}[E_{i_5}, \gamma_{25}, \delta_{25}][\alpha_{25}] \\ &\quad + \mathbb{C}[E_{i_5}, \gamma_{25}, \delta_{25}][\iota_{25}, \beta_5^{(5)}],\end{aligned}$$

and

$$\begin{aligned}R_{25} &= (O_{25A}, O_{25E}) + \mathbb{C}[\alpha_{25}, E_{i_5}, \gamma_{25}, \delta_{25}][u_{25}] + \mathbb{C}[\alpha_{25}, E_{i_5}, \gamma_{25}, \delta_{25}][\iota_{25}, \beta_5^{(5)}] \\ &= I_{25} + \mathbb{C}[u_{25}, \alpha_{25}, E_{i_5}, \gamma_{25}, \delta_{25}] + \mathbb{C}[\delta_{25}, \beta_5^{(5)}] + \mathbb{C}[\delta_{25}, \beta_5^{(5)}]\iota_{25}.\end{aligned}$$

6.5. **The case $N=7$.** Put $R_7 = \mathbb{C}[C_7, \alpha_7, \beta_7, \gamma_7, \delta_7]$ and

$$I_7 = (O_{7a}, O_{7b}, O_{7c}, O_{7d}, O_{7e}, O_{7f}).$$

We see

$$\mathbb{C}[C_7, \alpha_7, \beta_7] = (O_{7d}) \oplus \mathbb{C}[C_7, \beta_7] \oplus \mathbb{C}[C_7, \beta_7]\alpha_7,$$

$$\mathbb{C}[C_7, \beta_7, \delta_7] = (O_{7a}) + \mathbb{C}[\beta_7][C_7] + \mathbb{C}[\beta_7][\delta_7],$$

$$\mathbb{C}[C_7, \beta_7, \gamma_7, \delta_7] = (O_{7f}) \oplus \mathbb{C}[C_7, \beta_7, \delta_7] \oplus \mathbb{C}[C_7, \beta_7, \delta_7]\gamma_7,$$

$$\mathbb{C}[C_7, \beta_7]C_7\gamma_7 \subset (O_{7b}) + \mathbb{C}[C_7, \alpha_7, \beta_7],$$

and

$$\begin{aligned} R_7 &= (O_{7e}, O_{7c}) + \mathbb{C}[C_7, \beta_7][\alpha_7] + \mathbb{C}[C_7, \beta_7][\gamma_7, \delta_7] \\ &= I_7 + \mathbb{C}[C_7, \beta_7] + \mathbb{C}[C_7, \beta_7]\alpha_7 + \mathbb{C}[\beta_7, \delta_7] + (\mathbb{C}[C_7, \beta_7] + \mathbb{C}[\beta_7, \delta_7])\gamma_7 \\ &= I_7 + \mathbb{C}[C_7, \beta_7] + \mathbb{C}[C_7, \beta_7]\alpha_7 + \mathbb{C}[\beta_7, \delta_7] + \mathbb{C}[\beta_7, \delta_7]\gamma_7. \end{aligned}$$

7. MAIN RESULTS

For reader's convenience, we review basic facts on graded rings. We say a ring R is graded if R is decomposed into a direct sum of additive groups

$$R = \bigoplus_{k=0}^{\infty} R_k$$

such that $R_k R_l \subset R_{k+l}$ for all $k, l \geq 0$. In this paper, we only deal with the case $R_0 = \mathbb{C}$. For example, \mathbb{C} is graded as $\mathbb{C}_k = \{0\}$ for $k > 0$, and so is $\mathcal{M}(N)$ as $\mathcal{M}(N)_k = \mathcal{M}_k(N)$. Moreover, for a graded ring R and $n_1, \dots, n_r > 0$, we define $S = R[X_1, \dots, X_m]^{[n_1, \dots, n_m]}$ to be a ring $R[X_1, \dots, X_m]$ which is graded as $X_i \in S_{n_i}$. For given graded rings R and S , a ring homomorphism $f : R \rightarrow S$ is said to be graded if $f(R_k) \subset S_k$ for $k \geq 0$. In the sequel, every homomorphism is meant to be graded.

For a graded ring R with $\dim R_k < \infty$ for all k , let

$$H(R) = \sum_{k=0}^{\infty} (\dim R_k) t^k \in \mathbb{Z}[[t]]$$

be the Hilbert function of R . We see $H(\mathbb{C}) = 1$ and $H(R[X]^{[n]}) = H(R)(1 - t^n)^{-1}$, in particular, $H(\mathbb{C}[X]^{[n]}) = (1 - t^n)^{-1}$ and

$$H(\mathbb{C}[X, Y]^{[2,2]}) = \frac{1}{(1 - t^2)^2} = H(\mathcal{M}(4)).$$

$$H(\mathbb{C}[X, Y]^{[2,4]}) = \frac{1}{(1 - t^2)(1 - t^4)} = H(\mathcal{M}(2)),$$

$$H(\mathbb{C}[X, Y]^{[2,6]}) = \frac{1}{(1 - t^2)(1 - t^6)} = \sum_{k:\text{even}} \left(\left[\frac{k}{6}\right] + 1\right) t^k,$$

$$\begin{aligned}
H(\mathbb{C}[X, Y]^{[4,6]})t^4 &= \frac{t^4}{(1-t^4)(1-t^6)} \\
&= \frac{1}{(1-t^2)(1-t^4)} - \frac{1}{(1-t^2)(1-t^6)} \\
&= \sum_{k:\text{even}} \left(\left[\frac{k}{4} \right] - \left[\frac{k}{6} \right] \right) t^k \\
&= \sum_{k:\text{even}} \left(\left[\frac{k-4}{12} \right] + 1 - \delta_{12\mathbb{Z}+6}(k) \right) t^k \\
&= H(\mathcal{M}(1))t^4.
\end{aligned}$$

Then the first main theorem is stated as follows:

Theorem 1. *We have*

$$\begin{aligned}
\mathcal{M}(1) &\simeq \mathbb{C}[E_4, E_6]^{[4,6]}, \\
\mathcal{M}(2) &\simeq \mathbb{C}[C_2, \alpha_2]^{[2,4]} = \mathbb{C}[C_2, E_4]^{[2,4]}, \\
\mathcal{M}(4) &\simeq \mathbb{C}[C_2, \alpha_4]^{[2,2]} = \mathbb{C}[C_2, C_4]^{[2,2]}.
\end{aligned}$$

Proof. In §3.1 we have shown that the natural homomorphism

$$\mathbb{C}[E_4, E_6]^{[4,6]} \rightarrow \mathcal{M}(1)$$

is surjective. By comparing dimensions on both sides, we easily see that the homomorphism is bijective. Thus we obtain

$$\mathbb{C}[E_4, E_6]^{[4,6]} \simeq \mathcal{M}(1).$$

For $N = 2$ and 4 , the assertions can be shown in a similar way. \square

Suppose that R is a graded ring. If I is an ideal of R and $R = I \oplus S$ as \mathbb{C} -vector spaces, then for each k , we see $R_k = (I \cap R_k) \oplus (S \cap R_k)$. Moreover, if I is homogeneous, then R/I is naturally graded and we see

$$\dim(R/I)_k = \dim(R_k/(I \cap R_k)) = \dim(S \cap R_k).$$

In particular, if $O \in R_{2k} + R_k Y + Y^2$, then we have seen $R[Y] = (O) \oplus R \oplus RY$ in the previous section, and thus

$$H(R[Y]^{[k]}/(O)) = H(R)(1 + t^k).$$

We have the following results:

Theorem 2. *We have*

$$\begin{aligned}
\mathcal{M}(3) &\simeq \mathbb{C}[C_3, \alpha_3, \beta_3]^{[2,4,6]}/(O_3), \\
\mathcal{M}(5) &\simeq \mathbb{C}[C_5, \alpha_5, \beta_5]^{[2,4,4]}/(O_5), \\
\mathcal{M}(6) &\simeq \mathbb{C}[C_3^{(2)}, \alpha_6, \beta_6]^{[2,2,2]}/(O_6), \\
\mathcal{M}(8) &\simeq \mathbb{C}[C_4^{(2)}, \alpha_4, \alpha_4^{(2)}]^{[2,2,2]}/(O_8), \\
\mathcal{M}(9) &\simeq \mathbb{C}[C_3, \alpha_9, \beta_9]^{[2,2,2]}/(O_9).
\end{aligned}$$

Proof. In §3.4 we have obtained the natural surjective homomorphism

$$\mathbb{C}[C_3, \alpha_3, \beta_3]^{[2,4,6]} \rightarrow \mathcal{M}(3),$$

and we showed in §5.6 that it induces the homomorphism

$$\mathbb{C}[C_3, \alpha_3, \beta_3]^{[2,4,6]} / (O_3) \twoheadrightarrow \mathcal{M}(3).$$

We have

$$\begin{aligned} H(\mathbb{C}[C_3, \alpha_3, \beta_3]^{[2,4,6]} / (O_3)) &= \frac{1+t^4}{(1-t^2)(1-t^6)} \\ &= \sum_{k:\text{even}} ((\lfloor \frac{k}{6} \rfloor + 1) + (\lfloor \frac{k-4}{6} \rfloor + 1)) t^k \\ &= H(\mathcal{M}(3)), \end{aligned}$$

thus

$$\mathbb{C}[C_3, \alpha_3, \beta_3]^{[2,4,6]} / (O_3) \simeq \mathcal{M}(3).$$

Similarly, since

$$\begin{aligned} H(\mathbb{C}[C_5, \alpha_5, \beta_5]^{[2,4,4]} / (O_5)) &= \frac{1+t^4}{(1-t^2)(1-t^4)} = \frac{2}{(1-t^2)(1-t^4)} - \frac{1}{1-t^2} \\ &= H(\mathcal{M}(5)), \end{aligned}$$

$$H(\mathbb{C}[C_3^{(2)}, \alpha_6, \beta_6]^{[2,2,2]} / (O_6)) = \frac{1+t^2}{(1-t^2)^2} = H(\mathcal{M}(6)),$$

we may get the assertions. \square

Theorem 3. *We have*

$$\begin{aligned} \mathcal{M}(7) &\simeq R_7^{[2,4,4,6,6]} / I_7, \\ \mathcal{M}(10) &\simeq R_{10}^{[2,2,2,4,4]} / I_{10}, \\ \mathcal{M}(12) &\simeq R_{12}^{[2,2,2,2,2]} / I_{12}, \\ \mathcal{M}(16) &\simeq R_{16}^{[2,2,2,2,2]} / I_{16}. \end{aligned}$$

Proof. Since

$$\begin{aligned} R_7 &= I_7 \oplus \mathbb{C}[C_7, \beta_7] \oplus \mathbb{C}[C_7, \beta_7]\alpha_7 \oplus \mathbb{C}[\beta_7, \delta_7]\delta_7 \oplus \mathbb{C}[\beta_7, \delta_7]\gamma_7, \\ R_{10} &= I_{10} \oplus \mathbb{C}[u_{10}, \alpha_{10}] \oplus \mathbb{C}[\alpha_{10}, \beta_{10}]\beta_{10} \oplus \mathbb{C}[\beta_{10}, \zeta_{10}]\zeta_{10} \oplus \mathbb{C}[\beta_{10}, \zeta_{10}]\epsilon_{10}, \\ R_{12} &= I_{12} \oplus \mathbb{C}[C_3^{(2)}, \alpha_6] \oplus \mathbb{C}[\alpha_6, \beta_6]\beta_6 \oplus \mathbb{C}[\beta_6, \gamma_{12}]\gamma_{12} \oplus \mathbb{C}[\gamma_{12}, \beta_6^{(2)}]\beta_6^{(2)}, \end{aligned}$$

we see

$$\begin{aligned} H(R_7^{[2,4,4,6,6]} / I_7) &= \frac{1+t^4}{(1-t^2)(1-t^4)} + \frac{2t^6}{(1-t^4)(1-t^6)} \\ &= \frac{1+2t^4+t^6}{(1-t^2)(1-t^6)} \\ &= 2 \frac{1+t^4}{(1-t^2)(1-t^6)} - \frac{1}{1-t^2} \\ &= H(\mathcal{M}(7)), \end{aligned}$$

$$\begin{aligned}
H(R_{10}^{[2,2,2,4,4]}/I_{10}) &= \frac{1+t^2}{(1-t^2)^2} + \frac{2t^4}{(1-t^2)(1-t^4)} \\
&= \frac{2t^2}{(1-t^2)^2} + \frac{1+t^4}{(1-t^2)(1-t^4)} \\
&= H(\mathcal{M}(10)),
\end{aligned}$$

$$H(R_{12}^{[2,2,2,2,2]}/I_{12}) = \frac{1+3t^2}{(1-t^2)^2} = H(\mathcal{M}(12)),$$

and we get the assertions. \square

Theorem 4. *We have*

$$\mathcal{M}(18) \simeq R_{18}^{[2,2,2,2,2,2,2]}/I_{18},$$

$$\mathcal{M}(25) \simeq R_{25}^{[2,2,2,2,2,4,4]}/I_{25}.$$

Proof. Since

$$\begin{aligned}
R_{18} &= I_{18} \oplus \mathbb{C}[u_{18}, \alpha_{18}] \oplus \mathbb{C}[\alpha_{18}, \beta_6] \alpha_{18} \\
&\quad \oplus \mathbb{C}[\beta_6, \gamma_{18}] \beta_6 \oplus \mathbb{C}[\gamma_{18}, \delta_{18}] \gamma_{18} \oplus \mathbb{C}[\delta_{18}, \epsilon_{18}] \delta_{18} \oplus \mathbb{C}[\epsilon_{18}, \beta_6^{(3)}] \epsilon_{18}, \\
R_{25} &= I_{25} \oplus \mathbb{C}[u_{25}, \alpha_{25}] \oplus \mathbb{C}[\alpha_{25}, E_{i_5}] E_{i_5} \oplus \mathbb{C}[E_{i_5}, \gamma_{25}] \gamma_{25} \oplus \mathbb{C}[\gamma_{25}, \delta_{25}] \delta_{25} \\
&\quad \oplus \mathbb{C}[\delta_{25}, \beta_5^{(5)}] \beta_5^{(5)} \oplus \mathbb{C}[\delta_{25}, \beta_5^{(5)}] \iota_{25},
\end{aligned}$$

we see

$$H(R_{18}^{[2,2,2,2,2,2,2]}/I_{18}) = \frac{1+5t^2}{(1-t^2)^2} = H(\mathcal{M}(18)),$$

$$\begin{aligned}
H(R_{25}^{[2,2,2,2,2,4,4]}/I_{25}) &= \frac{1+3t^2}{(1-t^2)^2} + \frac{2t^4}{(1-t^2)(1-t^4)} \\
&= H(\mathcal{M}(25)),
\end{aligned}$$

and we get the assertions. \square

8. INTEGRALITY OF THE BASIS

We easily see that the basis $\{b_j\}$ taken in §3 is rational, namely, for each j

$$b_j \in q^{j-1} + \mathbb{Q}[[q]]q^j.$$

In this section, we would prove that those are integral, namely, for each j

$$b_j \in q^{j-1} + \mathbb{Z}[[q]]q^j.$$

Indeed, when $N = 1$, the assertion follows from $E_4, E_6, \Delta \in \mathbb{Z}[[q]]$. Once we may take such an integral basis $\{b_j\}$, we easily obtain that

$$\mathcal{M}_k(N) \cap (\mathbb{Z} \oplus \mathbb{Z}q \oplus \cdots \oplus \mathbb{Z}q^{d-1} \oplus \mathbb{C}[[q]]q^d) \subset \mathbb{Z}[[q]]$$

for each $k > 0$ and $N \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 16, 18, 25\}$, where $d = \dim \mathcal{M}_k(N)$.

8.1. **The case $N=4,6,8,9,12,16,18$.** We see

$$\tau_{N^2}(n) - \tau_N(n) = \sum_{d|n, N|d, N^2 \nmid d} d = N\tau_N\left(\frac{n}{N}\right),$$

where we make the convention that $\tau_N(\frac{n}{N}) = 0$ if $N \nmid n$. In particular, for each prime number p , we see

$$\frac{1}{p}((p+1)\tau_p - \tau_{p^2}) = \tau_p - \frac{1}{p}(\tau_{p^2} - \tau_p) = 1_p \cdot \sigma_1,$$

where $1_p = \delta_{\mathbb{N} \setminus p\mathbb{N}}$, since $\tau_p(pn) = \tau_p(n)$. Therefore we get

$$\alpha_4 = \sum_n \frac{1}{2}(3\tau_2 - \tau_4)(n)q^n = \sum_{n \equiv 1 \pmod 2} \sigma_1(n)q^n \in \mathbb{Z}[[q]],$$

$$\begin{aligned} \beta_9 &= \frac{1}{6} \left(\sum_n \frac{1}{3}(4\tau_3 - \tau_9)(n)q^n - E_{\rho_3} \right) \\ &= \sum_{n \equiv 2 \pmod 3} \frac{1}{3}\sigma_1(n)q^n \\ &= \sum_{n \equiv 2 \pmod 3} \sum_{d|n, d \equiv 1 \pmod 3} \frac{1}{3}(d + \frac{n}{d})q^n \in \mathbb{Z}[[q]], \end{aligned}$$

$$\begin{aligned} \gamma_{16} &= \sum_{n \equiv 3 \pmod 4} \frac{1}{4}\sigma_1(n)q^n \\ &= \sum_{n \equiv 3 \pmod 4} \sum_{d|n, d \equiv 1 \pmod 4} \frac{1}{4}(d + \frac{n}{d})q^n \in \mathbb{Z}[[q]]. \end{aligned}$$

We note that the integrality of γ_{16} can be proved also from

$$\gamma_{16} = (\alpha_4^{(2)} + 4\alpha_4^{(4)})\alpha_4/C_4^{(2)} \in \mathbb{Z}[[q]].$$

Moreover we easily reconfirm that our tuples taken in §3.1 are integral.

8.2. **The case $N=2,5,10,25$.** We see that our tuples taken in §3.2 are integral, for example,

$$\begin{aligned} \beta_5 &= \frac{1}{3} \left(-\sum_n \tau_5(n)q^n + \sum_n \sigma_3(n)(q^n - q^{5n}) + 20 \sum_n \sigma_3(n)q^{5n} \right) \\ &\quad - \left(\sum_n \tau_5(n)q^n \right)^2 \\ &= \sum_n \left(\sum_{d|n, 5 \nmid d} \frac{1}{3}(d^3 - d) + 48\sigma_3(\frac{n}{5}) \right) q^n - \left(\sum_n \tau_5(n)q^n \right)^2 \\ &\in \mathbb{Z}[[q]], \end{aligned}$$

$$\begin{aligned} \delta_{25} &= \frac{1}{100} \sum_n (6\tau_5(n) - \tau_{25}(n))q^n + \frac{1}{20}E_{\rho_5} - \frac{1}{10}E_{r_5} \\ &= \frac{1}{20} \sum_{5 \nmid n} (\sigma_1(n) + \rho_5(n)\sigma_1(n))q^n - \frac{1}{10}E_{r_5} \\ &= \frac{1}{10} \sum_{n \equiv 1 \pmod 5} (\sigma_1(n) - \sigma_{\rho_5}(n))q^n + \frac{1}{10} \sum_{n \equiv 4 \pmod 5} (\sigma_1(n) - \sigma_{\rho_5}(n))q^n \\ &= \sum_{n \equiv 1 \pmod 5} \sum_{d|n, d \equiv 2 \pmod 5} \frac{1}{5}(d + \frac{n}{d})q^n + \sum_{n \equiv 4 \pmod 5} \sum_{d|n, d \equiv 1 \pmod 5} \frac{1}{5}(d + \frac{n}{d})q^n \\ &\in \mathbb{Z}[[q]] \end{aligned}$$

8.3. **The case $N=3,7$.** We see that our tuples taken in §3.3 are integral, for example,

$$\begin{aligned}
\beta_7 &= \frac{1}{4} \left(- \sum_n \tau_7(n) q^n + \sum_n \sigma_3(n) (q^n - q^{7n}) + 30 \sum_n \sigma_3(n) q^{7n} \right) \\
&\quad - \frac{1}{2} \left(\sum_n \tau_7(n) q^n \right)^2 \\
&= \sum_n \left(\sum_{d|n, 7 \nmid d} \frac{1}{4} (d^3 - d) + 93 \sigma_3\left(\frac{n}{7}\right) \right) q^n - \frac{1}{2} \left(\sum_n \tau_7(n) q^n \right)^2 \\
&\in \mathbb{Z}[[q]] + \sum_n \sum_{d|n, 7 \nmid d} \frac{1}{4} (d^3 - d) q^n - \sum_n \frac{1}{2} \tau_7(n)^2 q^{2n} \\
&= \mathbb{Z}[[q]] + \sum_n \left(\sum_{d|n, 2|d, 7 \nmid d} \frac{1}{4} (-d) - \sum_{d|\frac{n}{2}, 7 \nmid d} \frac{1}{2} d^2 \right) q^n \\
&= \mathbb{Z}[[q]] + \sum_n \left(\sum_{d|\frac{n}{2}, 7 \nmid d} \frac{1}{2} (-d - d^2) \right) q^n \\
&= \mathbb{Z}[[q]],
\end{aligned}$$

$$\delta_7 = C_7^{-1} \beta_7^2 \in \mathbb{Z}[[q]].$$

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